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LAMINAR BOUNDARY-LAYER FLOWS WITH SMALL BUOYANCY EFFECTS

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Abstract—Approximate integral techniques are employed to investigate the effects of small buoyant forces on laminar flow over semi-infinite horizontal and vertical flat plates. Corrections to local shear stress and heat flux are presented for all Prandtl numbers from $Pr = 0{\cdot}01$ to $Pr = 1000$, and are seen to be in good agreement with previously computed "exact" values for $Pr > 0{\cdot}1$. The effect of buoyancy on the stability of the boundary layer is also computed.

NOMENCLATURE

- x, coordinate along plate measured positively from leading edge [ft];
- y, coordinate normal to plate measured positively away from plate [ft];
- u, v, fluid velocity components in x, y direction respectively [ft/s];
- T, fluid temperature [°R];
- u_∞, free stream velocity [ft/s];
- T_∞, free stream temperature [°R];
- β, volumetric coefficient of expansion [1/°R];
- g, gravitational acceleration [ft/s²];
- ν, kinematic viscosity [ft²/s];
- α, thermal diffusivity [ft/s];
- δ, mechanical boundary-layer thickness [ft];
- δ_T, thermal boundary-layer thickness [ft];
- η, dimensionless coordinate $= y/\delta$;
- η_T, dimensionless coordinate $= y/\delta_T$;
- $F(\eta)$, $G(\eta)$, dimensionless components of velocity profile;
- λ, velocity profile shape factor (dimensionless);
- $\theta(\eta_T)$, dimensionless temperature profile;
- ζ, dimensionless ratio of boundary-layer thicknesses $= \delta/\delta_T$;
- δ^*, dimensionless boundary-layer thickness $= \dfrac{\delta u_\infty}{\nu}$;
- x^*, dimensionless x coordinate $= \dfrac{x u_\infty}{\nu}$;
- Pr, Prandtl number $= \nu/\alpha$;
- Gr_x, Grashof number based on $x = \dfrac{g\beta(T_w - T_\infty)x^3}{\nu^2}$;
- Re_x, Reynolds number based on $x = \dfrac{x u_\infty}{\nu} = x^*$;
- ϵ, dimensionless measure of buoyancy effect $= Gr_x/Re_x^3$;
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, functions defined in equations (9), (10), (13) and (14);
- $\phi_1, \phi_2, \phi_3, \phi_4$, functions defined in equations (15) through (18);
- Re_c, Reynolds number based on distance from leading edge to point of instability;
- Re_{co}, Reynolds number based on distance from leading edge to point of instability with no buoyancy.

INTRODUCTION

ALTHOUGH the problems of free and forced convective heat transfer have been extensively treated separately, cases in which the two effects appear simultaneously have received relatively little analytical treatment. There are only a few recent works on this subject. Analyses for *small* buoyancy effects in laminar forced convective flow over a heated semi-infinite flat plate have been presented by Sparrow and Gregg [1] for a vertical plate, and by Mori [2] for a horizontal plate. Sparrow and Minkowycz [3] also considered the horizontal plate, pointing out an error of sign in Mori's work. The approach used by these authors has been to express the stream funtion $\psi(x, y)$ in terms of the usual Blasius similarity variable η or Pohlhausen similarity variable ξ, then expand ψ in a power series of a suitable small parameter about its value with no buoyancy effects present. The resulting differential equations for the higher order corrections to the stream function have been integrated numerically on an electronic computer for a few specific values of the Prandtl number.

In this short note the problems stated above are treated using the von Kármán–Pohlhausen integral technique. This technique has also been used by Gill and del Casal [4], and Acrivos [5]. Gill and del Casal formulated the approximate equations for a horizontal flat plate, but were not able to obtain any solutions. Acrivos derived the equations for the vertical flat plate and reported numerical results by using an electronic computer. Here it is found that for *small* buoyancy, closed-form solutions can be obtained which readily yield shear stress and heat flux corrections for any value of Prandtl number. Some observations on the stability of the laminar boundary layer are also made.

ANALYSIS

A. Horizontal plate

The equations of conservation of mass, momentum and energy for this case are given in [3] as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\frac{\partial}{\partial x}\int_y^\infty (T - T_\infty)\,\mathrm{d}y + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}\,. \tag{3}$$

In formulating these equations the usual boundary-layer approximations were made and all fluid properties were considered constant except the density, which was allowed to vary with temperature. A further approximation, common to treatments of buoyancy affected flows, restricts the effect of variable density to the formation of a "buoyant force" term, which is the first term appearing on the right-hand side of equation (2).

Integrating with respect to y from zero to infinity, and incorporating equation (1) into equations (2) and (3), the integral forms of the momentum and energy equations become

$$u_\infty^2\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)\mathrm{d}y = - g\beta\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty\left(\int_0^\infty \theta\,\mathrm{d}y\right)\mathrm{d}y + \nu\left(\frac{\partial u}{\partial y}\right)_0 \tag{4}$$

and

$$u_\infty\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \frac{u}{u_\infty}\theta\,\mathrm{d}y = -a\left(\frac{\partial u}{\partial y}\right)_0, \tag{5}$$

where $\theta = T - T_\infty/T_w - T_\infty$.

Following Pohlhausen, let us assume velocity and temperature profiles of the form

$$\frac{u}{u_\infty} = (2\eta - 2\eta^3 + \eta^4) + \frac{\lambda(x)}{6}(1 - \eta^3)\eta = F(\eta) + \lambda\,G(\eta) \tag{6a}$$

and

$$\theta = 1 - 2\eta_T + 2\eta_T^3 - \eta_T^4 = 1 - F(\eta_T) \tag{6b}$$

for η and η_T less than unity, and $u/u_\infty = 1$, $\theta = 0$ for η and η_T greater than unity respectively. The quantities η and η_T are given by $y/\delta(x)$ and $y/\delta_T(x)$, where δ and δ_T are the extent of the hypothetical mechanical and thermal boundary-layer thicknesses respectively. The expressions (6a) and (6b) conform to the normal Pohlhausen restrictions (Schlichting [6]), except that $\lambda(x)$ must be chosen to satisfy equation (2) at $y = 0$. This gives

$$\lambda(x) = \frac{g\beta(T_w - T_\infty)}{\nu u_\infty} \delta^2 \frac{d\delta_T}{dx} \int_0^\infty \theta(\eta_T)\, d\eta_T = \frac{3}{10} \frac{g\beta(T_w - T_\infty)}{\nu u_\infty} \delta^2 \frac{d\delta_T}{dx}.$$

If we introduce the dimensionless quantities $\zeta = \delta/\delta_T$, $\delta^* = u_\infty \delta/\nu$, $x^* = u_\infty x/\nu$ and substitute (6a) and (6b) into (4) and (5), we then obtain the following pair of coupled non-linear ordinary differential equations:

$$\frac{37}{315}\delta^* \frac{d\delta^*}{dx^*} - \frac{3}{9{,}450}\epsilon\delta^* \frac{d}{dx^*}\left[\delta^{*3} \frac{d}{dx^*}(\zeta\delta^*)\right] - \frac{1}{100{,}800}\epsilon^2\delta^* \frac{d}{dx^*}\left[\delta^{*5}\left(\frac{d\zeta\delta^*}{dx^*}\right)^2\right] = 2 + \epsilon\delta^{*2}\frac{d(\zeta\delta^*)}{dx^*}\left(\frac{1}{20} - \frac{4}{30}\zeta\right), \quad (7)$$

$$\zeta\delta^* \frac{d}{dx^*}\left\{\zeta\delta^*\left[H_1(\zeta) + \frac{3}{10}\epsilon\delta^{*2}\frac{d(\zeta\delta^*)}{dx^*} H_2(\zeta)\right]\right\} = \frac{2}{Pr}, \quad (8)$$

where

$$\epsilon = \frac{g\beta(T_w - T_\infty)\nu}{u_\infty^3} = \frac{Gr_x}{Re_x^3}$$

and $H_1(\zeta)$, $H_2(\zeta)$ are known polynomials of ζ (cf. Dienemann [7]). These polynomials are computed by performing the integrals

$$H_1(\zeta) = \int_0^1 F(\zeta\eta_T)\theta(\eta_T)\, d\eta_T \quad \text{for } \zeta < 1$$

$$\int_0^{1/\zeta} F(\zeta\eta_T)\theta(\eta_T)\, d\eta_T \quad \text{for } \zeta > 1$$

and

$$H_2(\zeta) = \int_0^1 G(\zeta\eta_T)\theta(\eta_T)\, d\eta_T \quad \text{for } \zeta < 1$$

$$\int_0^{1/\zeta} G(\zeta\eta_T)\theta(\eta_T)\, d\eta_T \quad \text{for } \zeta > 1.$$

The quantity ϵ is a measure of the ratio of buoyant forces to inertial forces. Equations similar to these were given in [4], however no solutions were obtained. For *small* buoyancy effects we assume that the solutions to (7) and (8) can be expressed as a power series in ϵ as follows:

$$\delta^* = \delta_0^* + \epsilon\delta_1^* + \ldots$$

$$\zeta = \zeta_0 + \epsilon\zeta_1 + \ldots .$$

Substituting these expressions for δ^* and ζ into (7) and (8) and collecting like powers of ϵ results in an elementary set of uncoupled ordinary differential equations which can be integrated immediately. The zero order solutions are

$$\delta_0^* = 2\left(\frac{317}{35}\right)^{1/2} x^{*\frac{1}{2}}, \quad \zeta_0 = \text{const.} = \zeta_0(Pr).$$

The dependence of ζ_0 on Pr may be determined from [7]. The first order solutions are

$$\delta_1^* = (1{\cdot}007\,\zeta_0 - 2{\cdot}207\,\zeta_0^2)\,\delta_0^* x^{*\frac{3}{2}} = \alpha_1(Pr)\,\delta_0^* x^{*\frac{3}{2}} \quad (9)$$

$$\zeta_1 = -\frac{\left[\frac{0{\cdot}352}{Pr}\alpha_1 + (59{\cdot}59)\,\zeta_0^3 H_2(\zeta_0)\right]}{\left[\frac{0{\cdot}352}{Pr} + 2\zeta_0^3 h(\zeta_0)\right]}\zeta_0 x^{*\frac{3}{2}} = \alpha_2(Pr)\,\zeta_0 x^{*\frac{3}{2}} \quad (10)$$

where

$$h(\zeta_0) = \frac{2}{15} - \frac{9}{140}\zeta_0^2 + \frac{1}{45}\zeta_0^3, \quad \zeta < 1$$

$$h(\zeta_0) = \frac{3}{10}\frac{1}{\zeta_0^2} - \frac{4}{15}\frac{1}{\zeta_0^3} + \frac{6}{70}\frac{1}{\zeta_0^5} - \frac{5}{180}\frac{1}{\zeta_0^6}, \quad \zeta > 1.$$

In these solutions we have imposed the initial conditions $\delta_0^*(0) = \delta_1^*(0) = 0$ and $\zeta_0(0) = \text{const.}$, $\zeta_1(0) = 0$. Since δ^* and ζ are only intermediate quantities in computing shear stress and heat flux, they are left in this form at this point.

B. *Vertical plate*

The governing equations for this case are given in [1]. Integrating these with respect to y as in case *A* we obtain

$$u_\infty^2 \frac{d}{dx} \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy =$$

$$- g\beta(T_w - T_\infty) \int_0^\infty \theta \, dy + \nu \left(\frac{\partial u}{\partial y}\right)_0$$

and

$$u_\infty \frac{d}{dx} \int_0^\infty \frac{u}{u_\infty} \theta \, dy = - \alpha \left(\frac{\partial \theta}{\partial y}\right)_0$$

As in case A we assume

$$\frac{u}{u_\infty} = F(\eta) + \lambda G(\eta), \quad \theta = 1 - F(\eta_T),$$

but now

$$\lambda = \frac{g\beta(T_w - T_\infty)}{\nu u_\infty} \delta^2.$$

The resulting differential equations for δ^* and ζ are

$$\left(\frac{37}{315} - \frac{1}{315} \epsilon\delta^{*2} - \frac{5}{9{,}072}\epsilon^2 \delta^{*4}\right) \delta^* \frac{d\delta^*}{dx^*} = 2 + \epsilon\delta^{*2}\left(\frac{1}{6} - \frac{3}{10}\zeta\right) \quad (11)$$

and

$$\zeta\delta^* \frac{d}{dx^*}\left\{\zeta\delta^*\left[H_1(\zeta) + \epsilon\delta^{*2} H_2(\zeta)\right]\right\} = \frac{2}{Pr}. \quad (12)$$

Similar equations have also been considered in [5], and were integrated numerically for any amount of buoyancy present.

For small buoyancy effects we can obtain solutions by expanding δ^* and ζ as in case A. Solving the resulting equations we obtain

$$\delta_0^* = 2\left(\frac{315}{37}\right)^{1/2} x^{*\frac{1}{2}}, \quad \zeta_0 = \text{const.} = \zeta_0(Pr)$$

as for the horizontal plate, and

$$\delta_1^* = (0{\cdot}940 - 1{\cdot}277\,\zeta_0)\,\delta_0^* x^* = \alpha_3(Pr)\,\delta_0^* x^*, \quad (13)$$

$$\zeta_1 = - \frac{\left[\frac{0{\cdot}470}{Pr}\alpha_3 + 102{\cdot}16\,\zeta_0^2 H_2(\zeta_0)\right]}{\left[\frac{0{\cdot}470}{Pr} + 3\zeta_0^3 h(\zeta_0)\right]} \zeta_0 x^* = \alpha_4(Pr)\zeta_0 x^*. \quad (14)$$

As in the case A, we have made the restrictions $\delta_0^*(0) = \delta_1^*(0) = 0$, and $\zeta_0(0) = \text{const.}$, $\zeta_1(0) = 0$.

C. Effect on stability of boundary layer

As was pointed out in [3], buoyancy effects induce longitudinal pressure gradients on a flat plate in an otherwise uniform flow. If the buoyancy effect is considered to manifest itself *only* in an induced pressure gradient and thereby affect the velocity profile, Schlichting's procedure for computing transition on airfoils [8] can be summarily adopted to compute the effect buoyancy has on the point of instability on a flat plate. From [8] we find that for small values of the quantity λ, the critical Reynolds number is given by $Re = 645 \exp(0{\cdot}6\lambda)$, so that the point of instability is reached when the Reynolds number based on boundary-layer displacement thickness reaches this critical value. This criterion yields

$$\delta^* \left[\int_0^\infty (1 - F)\, d\eta - \lambda \int_0^\infty G\, d\eta\right] = 645 \exp(0{\cdot}6\,\lambda).$$

For small values of λ the exponential is expanded in powers of ϵ. Inserting the proper expressions for δ^* and $\lambda(x)$, we obtain a homogeneous polynomial in x^*. The roots determine the point of instability and its dependence on the parameters (ϵ, Pr). Letting Re_c be the Reynolds number based on distance from the leading edge at the point of instability, and Re_{co} its value with no buoyance we find

$$\frac{Re_c}{Re_{co}} = 1 + \epsilon\,\{738(3{\cdot}07\,\zeta_0 - \alpha_1)\}$$

for the horizontal plate and

$$\frac{Re_c}{Re_{co}} = 1 + \epsilon\,\{(272{,}000)(35 - \alpha_3)\}$$

for the vertical plate.

RESULTS

Shear stress

The shear stress on the plate is given by

$$\tau = \frac{2\mu u_\infty}{\delta}\left(1 + \frac{\lambda}{12}\right).$$

For the horizontal plate, the ratio of the shear stress to its value with no buoyancy τ/τ_0 is given by

$$\tau/\tau_0 = 1 + \epsilon\left\{\frac{\zeta_0\delta_0^{*2}}{40}\frac{d\delta_0^*}{dx^*} - \frac{\delta_1^*}{\delta_0^*}\right\} =$$

$$1 + \epsilon x^{*\frac{1}{2}}\phi_1(Pr) = 1 + \frac{Gr_x}{Re_x^{5/2}}\phi_1(Pr) \quad (15)$$

and for the vertical plate by

$$\tau/\tau_0 = 1 + \epsilon\left[\frac{\delta_0^{*2}}{12} - \alpha_3\right] = 1 + \epsilon x^*\phi_3(Pr) =$$

$$1 + \frac{Gr_x}{Re_x^2}\phi_3(Pr). \quad (16)$$

The quantities ϕ_1 and ϕ_3 are shown in Fig. 1. The corresponding point values from [1] and [3] are shown for comparison.

Heat flux

The heat flux is given by

$$q = \frac{2k(T_w - T_\infty)}{\zeta\delta}.$$

For the horizontal plate, the ratio of the heat flux to its value with no buoyancy q/q_0 is given by

$$q/q_0 = 1 + \epsilon\left\{-\frac{\zeta_1}{\zeta_0} - \frac{\delta_1^*}{\delta_0^*}\right\} =$$

$$1 + \epsilon x^{*\frac{1}{2}}\phi_2(Pr) = 1 + \frac{Gr_x}{Re^{5/2}}\phi_2(Pr) \quad (17)$$

and likewise for the vertical plate

$$q/q_0 = 1 + \epsilon x^*[-\alpha_3 - \alpha_4] =$$

$$1 + \epsilon x^*\phi_4(Pr) = 1 + \frac{Gr_x}{Re_x^2}\phi_4(Pr). \quad (18)$$

The quantities ϕ_2 and ϕ_4 are also shown in Fig. 1.

Effect on boundary-layer stability

The relationship between Re_c and Re_{co} for the Prandtl number = 0·7 and 10 are shown in Fig. 2. It can be seen that high Prandtl number fluids are least affected by buoyancy although the actual variation with Pr is quite small in case B. For cases in which ϵ is positive (upper surface of heated horizontal plate, lower surface of cooled horizontal plate, upflow on a heated vertical plate, downflow on a cooled vertical plate) the boundary layer becomes more stable while for negative ϵ it is less stable.

As a concrete example, consider the case of air at 80°F flowing over a plate with a velocity of 5 ft/s. For a horizontal plate, the Reynolds number at the point of instability is increased by a factor of 1·27 for a temperature difference $T_w - T_\infty = 230$ degF ($\epsilon = 10^{-5}$), while for a vertical plate a temperature difference of only 2·3 degF ($\epsilon = 10^{-7}$) is needed to increase it by a factor of 1·96.

DISCUSSION

From Fig. 1 it can be seen that the approximate integral techniques yield an estimation of buoyancy effects on a laminar boundary layer that are in good agreement with values calculated from the exact equations by a digital computer, except for extremely low values of Prandtl number. For very low values of Pr, the thermal boundary layer becomes much larger than the mechanical boundary layer. Since the pressure at a point in the mechanical boundary layer is computed by integrating $\beta(T - T_\infty)$ up to the edge of the thermal layer, for low Pr the magnitude of the computed buoyancy effect is very sensitive to the assumed profile. On the other hand for high Pr, the thermal layer is much smaller than the mechanical layer, and one would expect the effect to be much less sensitive to the assumed profiles. The results shown in Fig. 1 may then be used with reasonable confidence for all values of Pr greater than 0·1. Strictly speaking, to make the perturbation technique of solution valid, values of ϵ should be limited to $\epsilon < 10^{-5}$ for the horizontal plate, and $\epsilon < 10^{-7}$ for the vertical plate.

In analyzing the results in Fig. 2, it must be remembered that buoyancy effects have been considered as influencing the velocity profiles only, and the subsequent stability calculations include only the effect of changes in these profiles. An exact analysis of the buoyancy effect would have to consider the effect of stratification of density in the boundary layer, and the subsequent momentum and energy disturbance equations would have to be considered simultaneously. The results shown can be considered only as engineering approximations of the true magnitude of effect involved.

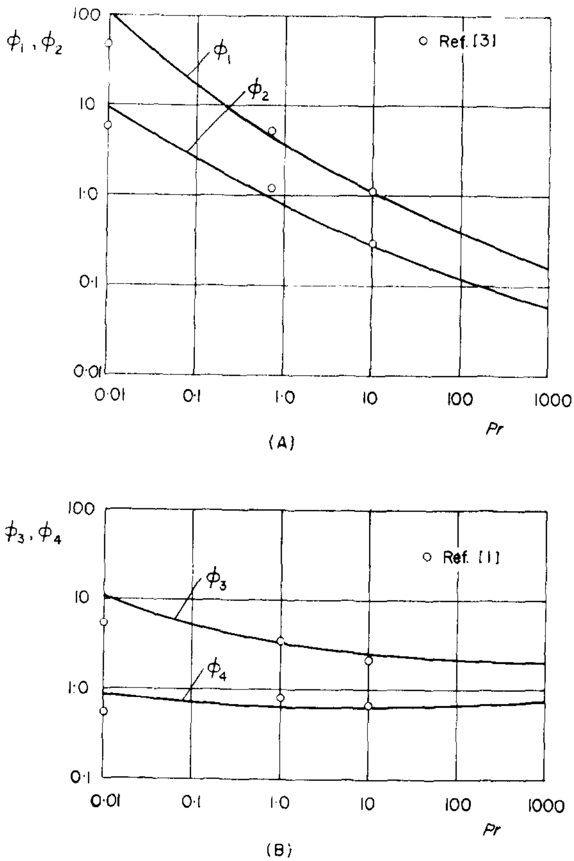

FIG. 1. The functions ϕ_1, ϕ_2, ϕ_3, ϕ_4, with which the corrections to shear stress and heat flux may be computed, versus Prandtl number *Pr*. (*A*) horizontal plate. (*B*) vertical plate.

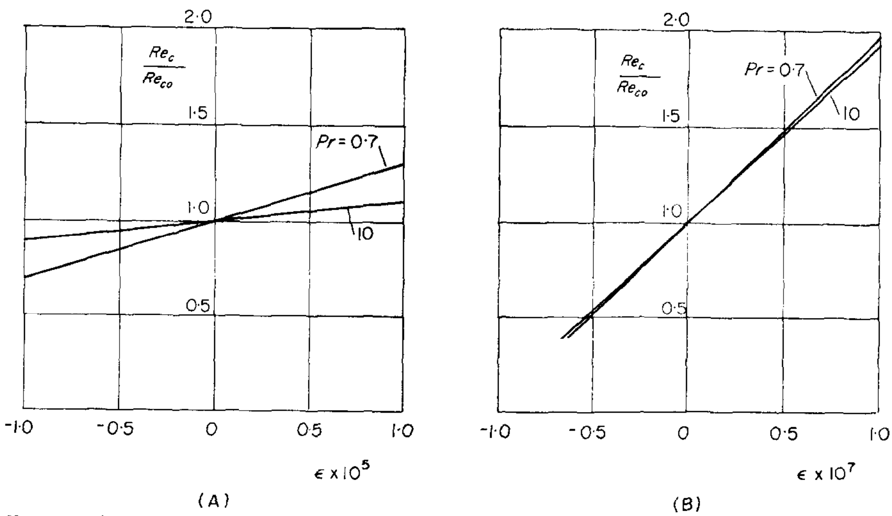

FIG. 2. The Reynolds number at the point of instability versus the measure of buoyancy $\epsilon = Gr_x/Re_x^3$. (*A*) horizontal plate. (*B*) vertical plate.

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Résumé—Des techniques intégrales approchées sont employées pour étudier les effets de forces d'Archimède faibles sur l'écoulement laminaire sur des plaques planes semi-infinies horizontales et verticales. Des corrections à la contrainte de cisaillement et au flux de chaleur locaux sont présentés pour tous les nombres de Prandtl allant de $Pr = 0{,}01$ à $Pr = 1000$, et l'on voit que l'accord est bon avec des valeurs "exactes" pour $Pr = 0{,}1$ calculées auparavant. L'effet des forces d'Archimède sur la stabilité de la couche limite est aussi calculé.

Zusammenfassung—Um den Einfluss geringer Auftriebskräfte auf die laminare Strömung über einer halbunendlichen waagerechten und senkrechten ebenen Platte zu untersuchen, wurde eine Näherungs-Integraltechnik verwendet. Für alle Prandtlzahlen von $Pr = 0{,}01$ bis $Pr = 1000$ werden Korrekturen für die lokale Schubspannung und die örtliche Wärmestromdichte angegeben. Sie weisen eine gute Übereinstimmung mit früher errechneten "exakten" Werten für $Pr > 0{,}1$ auf. Der Einfluss des Auftriebs auf die Stabilität der Grenzschicht wird ebenfalls berechnet.

Аннотация—Приближенные интегральные методы используются для исследования влияния малых сил плавучести на ламинарный поток на полубесконечной горизонтальной и вертикальной плоской пластинах. Даются поправки на локальное напряжение трения и плотность теплового потока для всех чисел Прандтля от $Pr = 0{,}101$ до $Pr = 1000$. Эти поправки хорошо согласуются с ранее вычисленными «точными» значениями для $Pr > 0{,}1$. Также определено влияние плавучести на стабильность пограничного слоя.